Eastern Illinois University

The Keep

Plan B Papers

Student Theses & Publications

1-1-1962

Intergration of Plane and Solid Geometry

Leo W. Cordes

Mathematics 570

Plan B term paper

by

Leo W. Cordes

Eastern Illinois University

INTEGRATION OF PLANE AND SOLID GEOMETRY

by

Leo W. Cordes

Dr. L. R. Van Deventer
Mathematics 570
Summer 1962

Integration of Plane and Solid Geometry

CHAPTER I

STATEMENT OF PURPOSE

Today with the rapid development of new branches of mathematics and the increased need for more people with good mathematical backgrounds in the scientific fields there is a general tendency to move more advanced mathematics down to the high school level. This gives rise to the need for reorganization of the high school mathematics courses. They must be reorganized with some of the material discarded in favor of other material if the high school is to keep up with the general trend.

Along with the increased interest in mathematics for use in science there is also a trend to introduce set theory early in the mathematics curriculum and to teach the traditional mathematics in a new integrated form. This modernization of the mathematics curriculum can be seen in the recent textbook publications of the major textbook companies.

In recent times solid geometry has not been taught in many high schools because of the small enrollments in these classes. There are some very useful concepts in solid geometry which should not be completely left out of the mathematics curriculum. The integration of plane and solid geometry would help toward the preservation of solid geometry in the high school curriculum.

Within the following pages of this paper it is my intention to arrive at some ways in which plane geometry and solid geometry can be combined to make a logical sequence without destroying the individuality of either of the two subjects. The ideas of plane geometry are in general less complex than the ideas of solid geometry and for the most part they will be introduced first and followed by corresponding ideas from solid geometry. This approach might in one sense be thought of as being a backward approach because in actuality solid geometry could encompass plane geometry.

It is traditional to teach plane geometry before solid geometry and I believe this to be the best procedure. In the following pages I will not attempt to put the ideas of solid geometry before those of plane geometry, but I will attempt to suggest some of the places in the traditional plane geometry course where the ideas of solid geometry can be inserted without losing the continuous aspect of the sequence. In one respect some of the solid geometry will come before some of the plane geometry, but the solid geometry will still follow the plane geometry as far as the different topics are concerned.

Plane geometry is generally recognized as accomplishing two main purposes. These would also apply equally as well to solid geometry. One of these purposes is to give the student a knowledge of the figures of the subject and the relationships which apply to these figures. In other words, this purpose is to give the student a knowledge of the facts of the subject which can be applied in solving problems.

The second main purpose is to develop within the student a knowledge of the logical processes of mathematics and how to use logical reasoning in other things as well as in mathematics. This aspect of the subject is found in the development of the proofs of the many theorems in geometry.

It is of course understood that in combining two courses into one course which is to be taught in the same length of time as the one course some of the subject matter will have to be omitted completely or all of it gone over lightly. I prefer to omit some of the more advanced topics of solid geometry and to use those parts which can be best inserted into the plane geometry sequence.

At the same time that some of the topics of solid geometry must be omitted some of the topics of plane geometry will have to be cut down in length and possibly some of the topics which usually appear at the end of the textbooks will have to be omitted.

With the above purposes in mind, it is my intention to develop a sequence of the subject matter of plane geometry and solid geometry which could be used as a course for sophomores in high school.

CHAPTER II

BASIC IDEAS OF GEOMETRY

When a student enters a geometry class he is already familiar with many of the forms and objects of geometry from his past experiences in school and out of school. These experiences can be used to form a basis for his further study of geometry. These experiences for the most part can be used mainly to develop his factual knowledge, but he also has had many experiences which might be utilized as a basis for the development of the logical processes of geometry.

For the beginning of the course in plane and solid geometry I would make use of the students prior experiences by discussing with the class such things as lines, points, triangles, angles, circles, squares, cubes, planes, cones, parallelograms, pyramids and spheres. This would be done on an informal basis by asking questions and directing the responses so that these things can be connected to the difference between what we call plane geometry and what we call solid geometry.

Models of solids such as cubes, cones and tetrahedrons would be used to show the nature of solid geometry and it would be pointed out that the faces of some of the figures are made up of triangles, squares and circles with which the student has already made an acquaintance. By pointing out the things with which the student already has some familiarity it is hoped that the student will feel at home with the subject and will be able to see its usefulness and desirability. With this accomplished the student will have the desire to go ahead and learn more about the subject.

Along with the introduction of a few of the basic forms of geometry would come a brief history of the subject. In this history would be the uses made of geometry by the ancient Greeks and Egyptians for such things as the measurement of land areas and for certain indirect measurements. This section would serve the purpose of showing the students that the subject of geometry is not new and has been very useful to man down through the many years of history.

An important aspect of any mathematical system is the notion of undefined terms. These are the very basis upon which the whole system to be developed depend and are thus very important. Certain terms in geometry are called undefined because they make use of ideas common to everyone, but which are very difficult to define without using these same terms to define themselves. They thus lead to circular definitions which are of no practical significance. We must have a starting place and in geometry the development of the system begins with the concept of certain undefined terms. Once we accept these terms we can then begin to build our system.

For the beginning of the formal part of the course in plane and solid geometry I would use as undefined terms the notions of point, straight line, surface and space. With these terms many other things could then be defined.

The teaching of the idea of undefined terms could be in the form of questions directed to the students to show the difficulty involved in trying to define things which we have taken for granted. When an apparent definition is given, the student might then be asked to define the terms he has used in the definition. This eventually will lead to a circular definition and thus point out the need for accepting certain terms without formal definitions for them.

After the acceptance of the idea of undefined terms, other terms can be introduced whose definitions can be given by using the undefined terms in defining them. As suggested by the words plane geometry, I believe one of the first definitions that should be given should be that of a plane. A plane could be defined as a surface which has the property that a straight line containing any two points on the surface lies completely on the surface.

After the definition of a plane would come the idea of plane figures and then the idea that if a figure is not a plane figure that it would be a solid geometry figure. Many other definitions will have to be introduced as the course progresses and the need arises for them.

The proof in geometry is used to develop logical reasoning. There are many types of proofs, but the one most used in geometry is the deductive proof. This type of proof consists of certain given information which along with axioms and postulates and previously proved theorems are used to deduce certain desired information.

"Any precise definition of proof is usually avoided. Certain mathematical disagreements arise because it has its basis in philosophy where there is a wide range of assumptions on which men fail to agree, and because entire books can well be devoted to defining terms and clarifying special cases. These problems can be avoided if 'the nature of proof' is discussed and if illustrations of some kinds of proof and analysis of some of the elements of proof are presented." [1]

To develop an understanding of proofs I would use the idea of congruence and introduce the concept of congruent triangles. This topic lends itself very usefully as an example for the development of proofs. Many individual exercises on congruent triangles should be used to show the variety of ways in which triangles can be proved congruent. While congruent triangles might be thought of as a classification of exercises, there are many individual aspects which can show variety.

As far as the nature of a proof is concerned it applies equally well to plane geometry and solid geometry. It should

---

[1] Lucien Blair Kinney and C. Richard Purdy, Teaching Mathematics in the Secondary School, New York: Rinehart and Company, Inc., 1952, p. 128.

be pointed out in the introduction of proofs that they apply to solid geometry as well as plane geometry and that two triangles do not need to be in the same plane in order to be congruent.

While the nature of a proof is being discussed other types of proof besides the deductive proof should be mentioned and illustrations of these given to the students. Indirect proofs are very useful for given situations and should not be overlooked, but I believe the direct deductive proof should be given preference in a beginning geometry course. The deductive proof lends itself better for an introductory course in logical reasoning.

CHAPTER III

PERPENDICULAR LINES AND PLANES

A section of the course in plane and solid geometry should be on the idea of perpendicularity. This topic affords an especially good opportunity to combine the ideas of plane and solid geometry. I believe there could be made a very logical connection between perpendicular lines and perpendicular planes.

The section on perpendicularity should begin with the defining of perpendicular lines. Illustrations of lines which are perpendicular could very easily be pointed out by the teacher and a contrast made between these lines and lines which are not perpendicular. The defining of perpendicular lines should be made in strict compliance with the previous definitions. This definition should be stressed as to how it relates to other definitions-- particularly as related to the definition of a right angle.

After the introduction and defining of perpendicular lines, use should be made of the perpendicular lines in the form of exercises. There should be given now some theorems about perpendicular lines. The teacher should now prove some of these theorems with the aid of the class and point the way toward further use of perpendicular lines.

When a good understanding is gained by the class on perpendicular lines in a plane, then I would introduce the notion of a line perpendicular to a plane. Using the definition of perpendicular lines in a plane the class could be asked to try to define what they think is meant by a line perpendicular to a plane. The teacher should direct the students toward a formal definition which would be useful for further development of the topic.

Along with the discovering of a suitable definition of a line perpendicular to a plane should be the use of models in three dimensions. These should be used to demonstrate the difference between lines which are perpendicular in one plane and a line which is perpendicular to a plane. Elaborate models are not necessary for this purpose. The teacher might simply use two pencils and the top of his desk for the demonstration. With the desk top representing a plane and the pencils representing lines the teacher could demonstrate relationships of lines and planes by changing the positions of the pencils.

There should then follow a discussion on the proving of lines perpendicular to planes along with further work and learning about the relations of planes to other planes. Perpendicular planes should be studied and theorems about them should be proved and learned by the students.

As a concluding section for the topic of perpendicularity I would choose the construction of perpendiculars given certain conditions. The first constructions would be those of perpendiculars in a plane. Included would be the construction of a line perpendicular to another line at a point on the line, through a point not on the other line, and the perpendicular bisector of a given line segment. It would then be pointed out that while we do not construct lines perpendicular to planes in the same manner in which we do constructions of plane figures, it is still possible to make drawings of lines and planes which appear to be as desired. Demonstrations of these should be done on the board by the teacher and the techniques acquired by the students. Practice in drawing planes and lines in three dimensions should then be done by the students in the form of assigned lessons.

CHAPTER IV

PARALLEL LINES AND PLANES

In the beginning of a section on parallel lines should be a definition of parallel lines. Usually parallel lines are defined as two lines in a plane which never meet however far they may be extended. A definition such as this is very difficult to use in a direct proof and should be used to prove some basic theorems which would be easier to use in direct proofs. An indirect proof is valuable in proving these basic theorems and an opportunity is afforded here for the acquaintance of the student with the indirect procedure. Theorems should be proved on the conditions for which lines are parallel and then these in turn used to prove lines parallel. For instance it could be proved from the definition that "Two lines in the same plane perpendicular to the same line are parallel." [2] This theorem could then be used to prove lines parallel.

A variety of exercises should be done by the student on parallel lines in a plane. These exercises with the direction of the teacher and the aid of models should give the student

---

[2]William C. Stone, Avery's Plane Geometry, Chicago: Allyn and Bacon, Inc., 1959, p. 111.

a knowledge and understanding of the concept of parallel lines.

Following the work on parallel lines in one plane would be the idea of three or more parallel lines which are not in the same plane. These could be easily demonstrated by the teacher pointing out lines in the classroom which are parallel, but not all in the same plane. The class could then be asked for further examples of parallel lines which are not all in the same plane.

With the introduction of parallel lines not in the same plane completed, the students could be asked to associate planes with pairs of parallel lines. This could lead to the conception of planes which are parallel. Parallel planes could then be defined and the intersections of planes which are not parallel could be examined to see if these intersections are in the form of any known concept. They should of course be associated with straight lines.

Several theorems and exercises should be done by the students. Theorems such as the following example could be used to associate parallel lines and parallel planes: "If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal and their planes are parallel." [3]

---

[3]William E. Kline, Robert Oesterle and Leroy M. Willson, Foundations of Advanced Mathematics, New York: American Book Company, 1959, p. 16.

CHAPTER V

POLYGONS AND POLYHEDRONS

A combined section of polygons and polyhedrons would be of great benefit for showing spatial relations and measurements of both plane and solid figures. It would offer great opportunity for the use of models for demonstration purposes. With models which are in three dimensions the students can better visualize areas and volumes. This section should help the student in his understanding of different kinds of measurements and give him further proof of the practical aspects of geometry.

"The aim of the work in solid geometry should be to exercise further the spatial imagination of the student and to give him both a knowledge of the fundamental spatial relationships and the power to work with them. It is felt that the work in plane geometry gives enough training in logical demonstration to warrent a shifting of emphasis in the work on solid geometry away from this aspect of the subject and in the direction of developing greater facility in visualizing spatial relations and figures, in representing such figures on paper, and in solving problems in mensuration." [4]

---

[4]John Wesley Young, Mathematics in Secondary Education, Chicago: Houghton Mifflin Company, 1927, p. 52.

I would begin this section with a study of the different kinds of polygons such as triangles, quadrilaterals, pentagons and hexagons. These should each be studied separately after an introduction to the general aspects of all polygons. There should be some use of proofs in this section, but the major objective here should be to teach the facts pertaining to the geometric figures.

To establish an order in the types of polygons to be studied I would teach them in the order of their number of sides, with the exception that the square would be used when giving the definition of area.

In the study of triangles there would be included isosceles, equilateral and scalene triangles. These should be compared and formulas for their areas and perimeters derived and used in a variety of numerical exercises.

Following the section on triangles would be a section on the different kinds of quadrilaterals. The rectangle and square would come first in this section and would be followed by the parallelogram and rhombus. The last quadrilaterals would be the trapezoid and the general quadrilateral. Formulas for all of the areas of these figures would be derived and many exercises would be given to develop facility in using the formulas. Aside from using the formulas the students should have a full understanding of the meanings of the formulas and should not make this section just a matter of memorizing formulas.

of pyramids, prisms and combinations of these would be included in this part of the work. Models of these should be available for the students to use and the teacher should make the distinction between area and volume plain to the students. Work with volumes would include some proofs, but they would have less emphasis in this section.

CHAPTER VI

RATIO, PROPORTION AND SIMILAR FIGURES

"We shall see that a ratio compares two quantities of the same kind by showing the quotient of the first divided by the second."[5] As we can see in this statement there is no limitation as to the types of quantities involved as long as they are both of the same kind. Ratio and proportion apply equally well to plane and solid figures.

While I feel that the ideas of ratio and proportion should be introduced with plane quantities there would seem to be no great barrier involved in extending ratio and proportion to quantities which are not in the same plane. This extension could involve ratios and proportions of line segments of solid figures, volumes and other measurements such as weights.

Similar figures would be introduced naturally after the ideas of ratio and proportion, because of the use made of these ideas. By the very definition of similar figures they are figures whose sides are proportional and they should logically follow the study of proportional line segments.

---

[5]William C. Stone, Avery's Plane Geometry, Chicago: Allyn and Bacon, Inc., 1959, p. 287.

The study of similar figures most logically would begin with similar triangles and progress to other plane figures. After the study of similar plane figures would come the study of solids which are similar. Exercises would be done by the students in which they would prove figures similar and in which they would derive numerical quantities from the proportions of the similar figures. Models could be used of similar solids and measurements taken on these for demonstration purposes.

For the practical aspects of similar figures the teacher could ask the students where they think that similar figures might be used. If there is no immediate response the teacher could start a discussion by mentioning such things as model airplanes and model trains. This should initiate the numeration of a long list of things which are made in the image of other things which are larger than themselves. The other direction might also be taken by mentioning such things as drawings or enlarged photographs of microscopic plants and animals. The teacher might mention, if none of the students do, the value of scaled blueprints to the architects and carpenters who design and build houses and other buildings.

CHAPTER VII

LOCUS

Greater difficulty is usually encountered by many students in the study of loci than in the study of the other theorems of geometry. This is probably due to the nature of locus theorems in which there are two parts to the proofs. In proving locus theorems it is not sufficient to prove that a set of points satisfy the given conditions, but it is necessary also to prove that there are no other points which also satisfy these conditions. It is usually difficult to get the students to see the difference between the two parts of the proof.

"Locus has a peculiar difficulty in that the concept has to be developed both in terms of restrictions on a point and of the path generated by a point; two propositions must be proved to establish a locus; and constructions and proofs involving locus present extremely varied problem-solving situations."[6]

Because of the double proof required in locus theorems, special care must be taken by the teacher in making clear the

---

[6]Lucien Blair Kinney and C. Richard Purdy, Teaching Mathematics in the Secondary School, New York: Rinehart and Company, Inc., 1952, p. 123.

reasons for this double proof. Illustrations of everyday locus problems could be given by the teacher as an introduction. The students might then be asked for illustrations from their own experiences and a common ground established.

Some basic constructions should be studied and reviewed early in the study of loci to establish the use made of loci in constructions.

Theorems from plane geometry on locus would be given before those of solid geometry because they are in general less difficult to prove and are easier to understand. The proofs of the locus theorems would be emphasized for the preliminary theorems, but would not be stressed so much for the more complicated theorems because of the time required.

Throughout the study of locus the teacher should stress and point out the value of loci in making constructions. He should also teach the value of sketches used to visualize a construction as it should be when it is completed. These sketches do not need to be in great detail, but ideas can very often be derived from them as to possible starting points for the construction.

CHAPTER VIII

CIRCLES, SPHERES AND SOLIDS WITH CURVED SURFACES

This section would begin with an informal discussion of circles, spheres, cylinders and cones. Since these have many applications in everyday life there should be no trouble in getting many examples of these from the students. Models can be examined and the differences of the shapes discussed.

Proofs of theorems in this section would be limited to circles. The main objective in this section would be to derive formulas for measurements of the different figures. An extensive study of the circle would precede the study of the solids. Formulas for the circle would include formulas for the circumference and area. For the solids, formulas for their total areas, lateral areas, and volumes would be studied and exercises involving numerical computations would be done by the students.

Constructions of circles and drawings of the solids would be a major part of this section. Drawings of the solids would be especially useful for the understanding of the numerical exercises. Without the drawings, problems can be very difficult to visualize sometimes.

While the concepts of circles and curved surfaces will not be exactly new to the students, they do possess rather different qualities than the straight lines and flat surfaces that they have been learning about. In the study of the circle the teacher should demonstrate the commonly known ratio of pi and show that it is the ratio of the circumference to the diameter of the circle. Estimations of the areas of some circles can be made to show that the formula for the area of a circle gives about what one would expect for the area in comparison with other figures.

A thorough study should be made of the angle relations in circles. Central angles, inscribed angles, angles formed by two intersecting chords and angles formed by two secants should be studied and exercises involving the sizes of these angles should be solved.

In the study of spheres it could be shown that if a plane was passed through the sphere the intersection of the plane and the sphere would be a circle. The largest of such circles would have its center at the center of the sphere. Some of the angle relations of the circle could thus be applied to the sphere. The volume of the sphere should be compared to the volume of a cube and some meaning thus applied to the formula for the volume of the sphere.

The volumes of cylinders and cones should be compared and a relation established between them. Once the relation has been established the student should have less difficulty in finding the volumes of cones.

CHAPTER IX

SUMMARY

With the modern trend toward the integration of the different subjects of mathematics and the better development of students of mathematics, the integration of plane and solid geometry is a logical step toward this end. The common aspects of the two subjects far out-weigh the differences and their integration can be made without too much difficulty.

In the preceding pages I have outlined a possible integrated course of plane and solid geometry. Many of the topics of solid geometry have their foundations in similar topics of plane geometry and I have attempted to relate similar topics in a logical fashion.

It is my feeling that the two main purposes of geometry can be satisfied as easily in an integrated course as in the two separate courses. These purposes being to teach the art of logical reasoning and the basic factual information of geometry.

BIBLIOGRAPHY

Kinney, Lucien Blair. Teaching Mathematics in the Secondary School, New York: Rinehart and Company, Inc., 1952.

Kline, William E.. Foundations of Advanced Mathematics, New York: American Book Company, 1959.

Stone, William C.. Avery's Plane Geometry, Chicago: Allyn and Bacon, Inc., 1959.

Young, John Wesley. Mathematics in Secondary Education, Chicago: Houghton Mifflin Company, 1927.